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# REMARKS ON PSEUDO-VALUATION RINGS

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**ABSTRACT.** A prime ideal $P$ of a ring $A$ is said to be a strongly prime ideal if $aP$ and $bA$ are comparable for all $a,b \in A$. We shall say that a ring $A$ is a pseudo-valuation ring (PVR) if each prime ideal of $A$ is a strongly prime ideal. We show that if $A$ is a PVR with maximal ideal $M$, then every overring of $A$ is a PVR if and only if $M$ is a maximal ideal of every overring of $A$ that does not contain the reciprocal of any element of $M$. We show that if $R$ is an atomic domain and a PVD, then $\dim(R) \leq 1$. We show that if $R$ is a PVD and a prime ideal of $R$ is finitely generated, then every overring of $R$ is a PVD. We give a characterization of an atomic PVD in terms of the concept of half-factorial domain.

## 1 INTRODUCTION

Throughout this paper, all rings are commutative with identity and the letter $R$ denotes an integral domain with quotient field $K$. Hedstrom and Houston

[11] introduced the concept *pseudo-valuation domains (PVD)*. Recall from [11] that an integral domain R, with quotient field K, is called a *pseudo-valuation domain* (PVD) in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. Recently, the author, Anderson, and Dobbs [8] generalized the study of pseudo-valuation domains to the context of arbitrary rings. From [8] a prime ideal P of a ring A is said to be a *strongly prime* ideal if aP and bA are comparable for all $a,b \in A$. If A is an integral domain, this is equivalent to the definition of strongly prime ideal introduced in [11] (see [3, Prop. 3.1], [4, Prop. 4.2], and [7, Prop.3]). We shall say that a ring A is a *pseudo-valuation ring (PVR)* if each prime ideal of A is a strongly prime ideal. For additional characterization of pseudo-valuation rings see [3], [4], [6], [7], and [8].

In this paper, we show that, for a PVR A with maximal ideal M, every overring of A ( inside its total quotient ring) is a PVR if and only if M is a maximal ideal of every overring of A that does not contain the reciprocal of any element of M. We show that if R is an atomic domain and a PVD, then $\dim(R) \leq 1$. We show that if R is a PVD and a prime ideal of R is finitely generated, then every overring of R is a PVD. We give a characterization of an atomic PVD in terms of the concept of half-factorial domain. Recall from Zaks [14] an atomic is called a *half-factorial domain (HFD)* if each factorization of a nonzero nonunit element of R into a product of irreducible elements (atoms) in R has the same length. Also, we give an alternative proof of the fact [2, Theorem 6.2] that an atomic PVD is a HFD.



## 2 RESULTS

We start by recalling some basic facts about a PVR.

**FACT 1 [ 8, Lemma 1].** (a). Let I be an ideal of a ring A and P be a strongly prime ideal of A. Then I and P are comparable.

(b). Any PVR is quasilocal.

**Proof. (a).** Suppose that I is not contained in P. Then for some $b \in I - P$ and $a = 1$, bA is not contained in $P = aP$, and so $P \subset bA \subset I$.

**(b).** This follows easily from (a). ∎

**Fact 2 [8, Theorem 2].** A quasilocal ring A with maximal ideal M is a PVR if and only if M is a strongly prime ideal.

The first part of the following result is taken from [7, Theorem 1] and the second part is a consequence of the above two Facts.

**LEMMA 3. (1).** If for each a,b in a ring A either $a|b$ or $b|a^2$, then the prime ideals of A are linearly ordered and therefore A is quasilocal.

(2). A ring A is a PVR if and only if it is quasilocal with its maximal ideal strongly prime.

**Proof.** (1). Suppose that there are two prime ideals P, Q of A that are not comparable. Let $b \in P \setminus Q$ and $a \in Q \setminus P$. Then neither $a|b$ nor $b|a^2$, a contradiction. (2). This follows easily from Facts 1 and 2. ∎

**DEFINITION.** Let b be an element of a ring B. Then an element d of B is called a proper divisor of b if $b = dm$ for some nonunit m of B.

In [8] ([7]) we proved that a ring A (R) is a PVR (PVD) if and only if for every $a, b \in A$ (R) either $a|b$ or $b|ac$ for each nonunit c of A (R). An analog of this result is the following proposition.

**PROPOSITION 4.** A ring B is a PVR if and only if for every $a, b \in B$, either $a|b$ or $d|a$ for every proper divisor d of b.

**Proof.** Suppose that B is a PVR with the maximal ideal M. Let $a, b \in B$ and suppose that a does not divide b in B. Let d be a proper divisor of b. Then $b = dm$ for some nonunit m of B. If d does not divide a in B, then $dM \subset aB$ since M is strongly prime. Hence, $a|dm = b$, a contradiction. Thus, $d|a$ for every proper divisor d of b.

Conversely, suppose that for every $a, b \in B$ either $a|b$ or $d|a$ for every proper divisor d of b. Let $a, b \in B$ such that a does not divide b in B. Then $b|a^2$, for otherwise by hypothesis $a|b$ which is a contradiction. Thus, by Lemma 3 (1) B is quasilocal with maximal ideal M. Now, we need show that $aM \subset bB$. Deny. Then there is a nonunit c of B such that b does not divide ac. Since a is a proper divisor of ac and b does not divide ac, by hypothesis $a|b$ which contradicts the assumption that a does not divide b in B. Hence, our denial is invalid. ∎



Anderson and Mott [2, Theorem 6.2] proved that an atomic PVD R is a HFD. Now, we give a proof of this result that relies only on the definitions of a PVD and a HFD.

**THEOREM 5 [2, Theorem 6.2].** An atomic PVD R is a HFD.

**Proof.** Deny. Let M be the maximal ideal of R. Then for some nonunit nonzero element x of R, $x = x_1x_2\ldots x_n = y_1y_2\ldots y_m$ where the $x_i$'s and the $y_j$'s are atoms of R and $m > n$. Hence, $(x_1/y_1)\ldots(x_n/y_n) = y_{n+1}\ldots y_m \in M$. Hence, for some i, $1 \le i \le n$, $x_i/y_i \in M$. Thus, $x_i = y_i m$ for some $m \in M$. A contradiction, since $x_i$ is an atom of R and neither $y_i$ nor m is a unit of R. Hence, our denial is invalid and R is indeed a HFD. ∎

**Definition.** Let R be a HFD and x be a non-zero element of R. Then we define $L(x) = n$ if $x = x_1x_2\ldots x_n$ for some atoms $x_i$, $1 \le i \le n$, of R. If x is a unit of R, then $L(x) = 0$.

In the following theorem, we give a characterization of an atomic PVD in terms of the concept of HFD.

**THEOREM 6.** Let R be an atomic domain. The following statements are equivalent :

(1) R is a PVD.

(2) R is a HFD and for ever $x,y \in R$, if $L(x) < L(y)$, then $x|y$ in R.

**Proof.** (1) ⇒ (2). By theorem 1 R is a HFD. Let $x,y \in R$ such that $L(x) < L(y)$. Suppose that x does not divide y in R. Then $y|xt$ for some atom t of R by [8, Prop. 3]. Hence, $xt=ym$ for some nonunit m of R (observe that if m is a unit of R, then $x|y$). But $L(xt) < L(ym)$, a contradiction, since R is a HFD. Thus, $x|y$. (2)⟹(1). Let $a,b \in R$ and suppose that a does not divide b in R. Then $L(b) \le L(a)$ by the hypothesis. Hence, $L(b) < L(ac)$ for every nonunit c of R. Thus, $b|ac$ for every nonunit c of R. Therefore, R is a PVD by [8, Prop.3]. ■

**COROLLARY 7 .** Let R be an atomic PVD, c is an atom of R, and $x \in R$. If $L(x) = n \ge 2$, then $x = c^{(n-1)}b$ for some atom b of R.

**Proof.** By Theorem 3, $c^{(n-1)}|x$ since $L(c^{(n-1)}) < L(x)$. Hence, $x = c^{(r-1)}b$ for some $b \in R$. Since R is a PVD, R is a HFD. Hence, b must be an atom of R. ■

Hedstrom and Houston [11] proved that a Notherian PVD R has a Krull dimension ≤ 1. We strengthen



this result in the next theorem. Before stating the following theorem, the following fact is needed :

**FACT 8 [ 7, Corollary 1 ].** Suppose that the prime ideals of a ring A are linearly ordered and a,b are nonzero elements of A. Let P be the minimum prime ideal of A that contains a and Q be the minimum prime ideal of A that contains b. Then P = Q if and only if there exist $n \ge 1$ and $m \ge 1$ such that $a|b^n$ and $b|a^m$. ■

**THEOREM 9.** Let R be an atomic PVD. Then $\mathrm{Dim}(R) \le 1$.

**Proof.** Let a,b be nonzero nonunit elements of R. By the above Fact, it suffices to show that $a|b^n$ for some $n \ge 1$. Let $m = L(a)$ and $h = L(b)$. Then for some $n \ge 1$, $m < nh$, that is, $L(a) < L(b^n)$. Hence, by Theorem 6 $a|b^n$. ■

**REMARK** : Anderson and Mott [2, Corollary 5.2] proved that R is an atomic PVD with maximal ideal M if and only if $V = M:M = \{ x \in K : xM \subset M \}$ is a discrete valuation domain with maximal ideal M. Since V and R have the same maximal ideal, by [5, Theorem 3.10] the prime ideals of V are the prime ideals of R. Hence, $\mathrm{Dim}(R) \le 1$ and this is another proof of Tneorem 9.

Recall that a domain R is a LT-domain (lowest terms domain ) in the sense of [1], if for each nonzero elements $a, b \in R$, there are nonzero elements $c, d$ of $R$ with $a/b = c/d$ and $\gcd(c,d) = 1$.

**COROLLARY 10.** Let $R$ be an atomic PVD. Then $R$ is a LT-domain.

**Proof.** Let $a, b$ be nonzero elements of $R$. We consider three cases. *First case.* Suppose that $L(a) < L(b)$. Then $a|b$. Hence, $a/b = 1/s$ for some $s \in R$ and $\gcd(1,s) = 1$. *Second case.* Suppose that $L(a) > L(b)$. Then $a/b = s/1$ for some $s \in R$ and $\gcd(s,1) = 1$. *Third case.* Suppose that $L(a) = L(b)$. Then $a = vh$ for some atom $h$ of $R$ and $v \in R$. Since $L(v) < L(b)$, $b = vd$ for some $d \in R$. Since $L(b) = L(a) = L(v) + 1$, $d$ is an atom of $R$. Hence, $a/b = h/d$. Since $h, d$ are atoms of $R$, $\gcd(h,d) = 1$. ■

In view of the proof of the above Theorem we have

**COROLLARY 11.** Let $R$ be an atomic PVD and $x = a/b \in K$ where $a, b$ are nonzero elements of $R$. Then $x = a/b$ must equal to one of the following forms :

(1) $1/s$ for some $s \in R$.

(2) $s/1$ for some $s \in R$.

(3) $h/d$ for some atoms $h, d$ of $R$.

**Definition.** For a ring $A$, let $S = \{ s \in A : s$ is a non-zerodivisor of $A$, that is, $s$ is regular $\}$. Then $T = R_S$ is the total quotient ring of $A$. A subring $B$ of $T$ is called an *overring* of $A$ if



**LEMMA 12.** Let $P$ be a strongly prime ideal of a ring $A$ containing the zerodivisors of $A$ and $B$ be an overring of $A$. If $x = a/b \in B\backslash A$ for some $a, b$ of $A$, then $a$ is a nonzerodivisor of $A$ and $x^{-1}P \subset P$.

**Proof.** Suppose that $a$ is a nonzerodivisor of $A$. Then $b \in P$, for if $b \in A\backslash P$, then $P \subset (b)$ and hence $b|a$ and therefore $x \in A$, a contradiction. Since $P$ is a strongly prime ideal, either $aA \subset bP$ or $bP \subset aA$. If $aA \subset bP$, then $b|a$ and therefore $x \in A$, a contradiction. If $bP \subset aA$, then $a|b^2$ since $b \in P$, a contradiction again, since $b$ is a nonzerodivisor of $A$ and $a$ is a zerodivisor of $A$. Hence, $a$ is a nonzerodivisor of $A$. Now, since $x = a/b \in B\backslash A$, $bP \subset aA$. Thus, $x^{-1}P \subset A$. Suppose that for some $p \in P$, $x^{-1}p = q \in A\backslash P$. Then $q$ is a nonzerodivisor of $A$, since $P$ contains the zerodivisors of $A$. Since $P$ is a strongly prime ideal and $qA \not\subset P$, $p \in P \subset qA$, and therefore $x = p/q \in A$, a contradiction. Hence, $x^{-1}P \subset P$. ■

The following Theorem is an important tool for the remaining part of this paper.

**THEOREM 13.** Let $P$ be a strongly prime ideal of a ring $A$ containing the zerodivisors of $A$ and B be an overring of $A$. The following statements are equivalent :

(1) $PB \cap A = P$.

(5) Every overring of A that does not contain the reciprocal of any element of M is a PVR.
(6) Every overring of A that does not contain the reciprocal of any element of M is a PVR with maximal ideal M.
(7) M is a maximal ideal of every overring C of A such that $C \subset M:M$.
(8) M is the unique maximal ideal of every overring C of A such that $C \subset M:M$.
(9) $A \subset C$ satisfies the INC condition for every overring C of A such that $C \subset M:M$. ■

## 3 EXAMPLES

**EXAMPLE 1.** Choose an infinite dimensional valuation ring (chained ring) V of the form $V = K + M$ where K is a field and M is the maximal ideal of V (see [10, Exercise 12, page 271]). If F is a proper subfield of K and [K:F] is infinite, then $R = F + M$ is a PVD (see [11, Example 2.1]). Observe that R has infinite Krull dimension and therefore is not atomic by Theorem 9.

**EXAMPLE 2.** Let $R = Z[\sqrt{5}]_{(2,\,1+\sqrt{5})}$. Then R is a Noetherian PVD and therefore atomic (see [11, Example 3.6]).

**EXAMPLE 3.** Let k be any field and X,Y be indeterminates. Then $R = k + Xk(Y)[[X]]$ is an atomic PVD that is not Noetherian (see the discussion in [2] following [2, Theorem 5.4]).



**EXAMPLE 4.** **(a)** Let R be a PVD. If I is an ideal of R, then R/I is a PVR by [8, Corollary 3].
**(b)** Let k be any field and x,y indetrminates. Then $R = k[X,Y]/(X^2,XY,Y^2)$ is a PVR ( see [8,Example 10]).

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# ON CHAINED OVERRINGS OF PSEUDO-VALUATION RINGS

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**ABSTRACT.** A prime ideal P of a commutative ring R with identity is called strongly prime if aP and bR are comparable for every a, b in R. If every prime ideal of R is strongly prime, then R is called a pseudo-valuation ring. It is well-known that a (valuation) chained overring of a Prufer domain R is of the form $R_P$ for some prime ideal P of R. In this paper, we show that this statement is valid for a certain class of chained overrings of a pseudo-valuation ring.

## 1. INTRODUCTION

Throughout this paper, all rings are commutative with identity and if R is a ring, then Z(R) denotes the set of zerodivisors of R and T denotes the total quotient ring of R. We say a ring A is an overring of a ring R if A is between R and T. Recall that a ring R is called a chained ring if the principal ideals of R are linearly ordered, that is, if for every $a, b \in R$ either $a|b$ or $b|a$. It is well-known that a chained overring of a Prufer domain R